Seshadri constants on rational surfaces with anticanonical pencils

Sano, Taro

代数幾何学シンポジウム記録
2009年度 pp.120-120

Seshadri constants on rational surfaces with anticanonical pencils

Taro SANO
tarosano@ms.u-tokyo.ac.jp
The university of Tokyo, Japan

Abstract: We provide an explicit formula for Seshadri constants of any polarizations on rational surfaces X such that dim $|\ \ K_X|\ \ 1$. As an application, we discuss relationship between singularities of log del Pezzo surfaces and Seshadri constants of their anticanonical divisors.

1 Def. & Motivation

Def. X: proj. var., L: ample l.b. on X, $x \in X$.
Seshadri constant of L at x is

$$(L, x) := \inf \ \frac{L\ \ C}{\mathrm{mult}_x(C)}; C \quad X : \text{irred. curve} \ni x$$

$$\overset{\text{well-known}}{=} \max\{s \in \mathbb{R};\ _x(L) \quad sE_x \text{ is nef}\}$$

where $_x : \widetilde{X}(x) \to X$: blow-up at x, $E_x := \ _x^{\ 1}(x)$: excep. div..

1st properties

Fix L. Then,

1. $_{\inf}(L) := \inf_{x \in X} \ (L, x) > 0$ (by Seshadri's ampleness criterion)
2. $_L : X \to \mathbb{R}; x \mapsto \ (L, x)$ takes constant value $_{\mathrm{gen}}(L)$ at very general points ($_L$: lower semiconti. if X is smooth.).

Ex. $X = \mathbb{P}^n, L = \mathcal{O}_{\mathbb{P}^n}(k) \Rightarrow \ (L, x) = k$ $(\forall x \in X)$.

Seshadri constants are related to various topics, e.g.;

- generation of jets of adjoint bundles $K_X + L$
- splitting of Abelian varieties into product
- Castelnuovo-Mumford regularity

Unfortunately, Seshadri constants are very hard to compute, even for rational surfaces.

Nagata conj. $r \quad 10$, $_r : X_r \to \mathbb{P}^2$: blow-up at $x_1, \ldots, x_r \in \mathbb{P}^2$: very general. Put $E_i^{(r)} := \ _r^{\ 1}(x_i)$. Then,

$$\Rightarrow \ _r\mathcal{O}_{\mathbb{P}^2}(1) \quad \frac{1}{\sqrt{r}} \sum_{i=1}^{r} E_i^r : \text{nef}$$

$$\Uparrow$$

$$\underline{\textbf{Conj}} \quad _{\mathrm{gen}} \quad _{r\ \ 1}\mathcal{O}_{\mathbb{P}^2}(1) \quad \frac{1}{\sqrt{r}} \sum_{i=1}^{r\ \ 1} E_i^{(r\ \ 1)} \Bigg) = \frac{1}{\sqrt{r}}.$$

2 An explicit formula for "special" rational surfaces

$_r : X_r \to \mathbb{P}^2$: blow-up at $x_1, \ldots, x_r \in \mathbb{P}^2$.

- If $x_1, \ldots, x_r \in \mathbb{P}^2$: very general $\Rightarrow |\ \ K_X| = \emptyset$.(Di culty)
- If $x_1, \ldots, x_r \in \mathbb{P}^2$: "special" $\Rightarrow$ dim $|\ \ K_X| \ \ 1 \Rightarrow$ easy to compute $\ (L, x)$

Main Thm. $\ : X \to \mathbb{P}^2$: blow-up at $x_1, \ldots, x_r \in \mathbb{P}^2$ s.t. dim $|\ \ K_X| \ \ 1$. $L := \ \mathcal{O}_{\mathbb{P}^2}(a) \quad \sum_{i=1}^r b_i E_i$: ample on X.

1. If $r = 9$ and $\sum_{i=1}^r b_i + \sqrt{a^2 \quad \sum_{i=1}^r b_i^2} \quad 3a$, then
$$_{\mathrm{gen}}(L) = 3a \quad \sum_{i=1}^{r} b_i$$
2. Otherwise, $\exists M_L > 0$ s.t
$$_{\mathrm{gen}}(L) = \min \ \frac{\alpha a \quad \sum_{i=1}^r \beta_i b_i}{\beta_{r+1}}; (\alpha; \beta_1, \ldots, \beta_{r+1}) \in \mathbb{Z}^{r+2}, \alpha \quad M_L,$$
$$\alpha^2 \ \sum_{i=1}^{r+1} \beta_i^2 = \ \ 1, 3\alpha \ \sum_{i=1}^{r+1} \beta_i = 1, \beta_{r+1} \quad 1 \Big\}$$

$\Uparrow$ + some calculation

Key Fact Let X be a rational surface s.t. $|\ \ K_X| \neq \emptyset$. Then

1. $\overline{NE}(X) = \mathbb{R}_{\ 0}[\ \ K_X] + \sum_{\substack{C \subset X:\text{irred. curve,} \\ C^2 < 0}} \mathbb{R}_{\ 0}[C]$.
2. $C \quad X$: irred. curve s.t. $C^2 < 0 \Rightarrow C$: (1)-curve, (2)-curve, or fix. comp. of $|\ \ K_X|$.

3 Application: criterion for mildly singular del Pezzo surfaces via $_{\mathrm{gen}}(\ \ K_X)$

X: log del Pezzo surf. i.e. normal klt proj. surf. s.t. $\ K_X$:ample. Assume X has $\mathbb{Q}$-Gorenstein smoothing $f : \mathfrak{X} \to T \ni 0$ with $\mathfrak{X}_0 \simeq X$. By lower semiconti. of Seshadri constants in family,

$$_{\mathrm{gen}}(\ \ K_{\mathfrak{X}_0}) \quad _{\mathrm{gen}}(\ \ K_{\mathfrak{X}_t}) \quad (\star).$$

Assume $4 \quad K_X^2 \quad 9$. We can determine when the equality hold in the inequality $(\star)$.

Theorem. X: a log del Pezzo surface.

1. Assume $K_X^2 = 9$. Then
$$_{\mathrm{gen}}(\ \ K_X) = 3 \Leftrightarrow X \simeq \mathbb{P}^2.$$
2. Assume $K_X^2 = 4, 5, 6, 7$ or 8. Then
$$_{\mathrm{gen}}(\ \ K_X) = 2 \Leftrightarrow \begin{cases} X \text{ has only Du Val sing.} \\ \quad \textit{or} \\ X \simeq Z(\ _5) \end{cases}$$
where $_5$ is a partition of 5 i.e.
$$_5 = (5), (1,4), (2,3), (1,1,3), (1,2,2), (1,1,1,2), (1,1,1,1,1).$$